

On the Topological Properties of the One Dimensional Exponential Random Geometric Graph

Bhupendra Gupta*, Srikanth K. Iyer[†] and D. Manjunath[‡]

Abstract

In this paper we study the one dimensional random geometric (random interval) graph when the location of the nodes are independent and exponentially distributed. We derive exact results and limit theorems for the connectivity and other properties associated with this random graph. We show that the asymptotic properties of a graph with a truncated exponential distribution can be obtained using the exponential random geometric graph.

Keywords: Random geometric graphs, exponential random geometric graphs, connectivity, components, degree, largest nearest-neighbor distance.

*Department of Mathematics and Statistics, Indian Institute of Technology, Kanpur 208016 INDIA. email: bhupen@iitk.ac.in.

[†]Department of Mathematics, Indian Institute of Science, Bangalore 560012 INDIA. email: skiyer@math.iisc.ernet.in

[‡]Department of Electrical Engineering, Indian Institute of Technology, Bombay, Mumbai 400076, INDIA. email:dmanju@ee.iitb.ac.in.

1 Introduction

We consider random geometric graphs (RGGs) in one dimension, $G_n(\lambda, r)$, with vertex set $V_n = \{X_1, \dots, X_n\}$ and edge set $E = \{(X_i, X_j) : |X_i - X_j| \leq r\}$, where X_i are i.i.d. exponential with mean λ^{-1} and r is called the “cutoff range”. Here X_i is used to denote the i^{th} vertex and its location. $G_n(\lambda, r)$ will be called an exponential RGG. We derive formulas and recursive algorithms when the number of nodes n and cutoff r are fixed. We then derive asymptotic results for the probability of connectivity, and weak law results for the number of components, total uncovered area etc. Strong law asymptotics are derived for the connectivity and largest nearest neighbor distances. We also obtain strong law results when the X_i are i.i.d. truncated exponential.

1.1 Previous Work and Background

The topological properties of RGGs have applications in wireless communication and sensor networks (e.g., [12]), cluster analysis (e.g., [9, 10]), classification problems in archaeological findings, traffic light phasing, and geological problems (e.g., [11]), and also in their own right (e.g., [17]).

The following are some results motivated by random wireless networks. For a network of n nodes distributed uniformly inside the unit circle, [12] obtains the asymptotic threshold function for the critical transmission range to be $\sqrt{\frac{\log n}{n}}$. More precisely, they show that with cutoff $r_n = \sqrt{\frac{\log n + c_n}{n}}$, the graph on n uniform points in the unit circle will be connected with probability approaching one iff $c_n \rightarrow \infty$. A result that enables the nodes to control local topological properties and work towards obtaining a connected network is derived in [22]. Here it is shown that for a static network with n nodes uniformly distributed over the unit circle, if each node is connected to $(5.1774 \log n)$ nodes, then the network is asymptotically connected. This problem has also been studied empirically in the context of multi-hop slotted Aloha networks [14]. The transmission radius for connectivity of a network when the placement of the nodes follows a Poisson process in dimensions $d \leq 2$ is derived in [6].

The following is a sample of the results from the study of RGGs in their own right. When n nodes are uniformly distributed in the d -dimensional unit cube, the following is shown in [16] for any l_p metric. Start with isolated points and keep adding edges in order of increasing length. Then, with a very high probability, the resulting graph becomes $(k + 1)$ -connected at the same edge length $r^*(n)$ at which the minimum degree of the graph becomes $k + 1$, for $k \geq 0$. With $k = 0$, this result means that the graph becomes connected with high probability at the same time that the isolated vertices disappear from the graph. [1, 2] is a similar study for the l_∞ norm. The best introduction to the study of RGGs via their asymptotic properties is [17].

Observe that the results cited above are all for the asymptotic case with $n \rightarrow \infty$. Exact

analysis of finite networks is important because the asymptotes may be approached very slowly. Exact analysis of finite networks have been considered in [7, 8, 11]. The probability of a connected network when n nodes are uniformly distributed in $[0, 1]$, with all nodes having the same transmission range r was derived in [7, 11]. In [11], algorithms for various connectivity properties of a one dimensional RGG with uniform distribution of nodes on the unit interval are derived. The probability of a specified labeled subgraph with edge set $E = \{(X_i, X_j) : |X_i - X_j| \leq w_{i,j}\}$ was obtained in [8]. This is then used to calculate exact probabilities for many network properties. This generalization dispenses with the requirement that the cutoff range be the same for all node pairs.

While the results described above are all for the case of an RGG when the nodes are distributed uniformly in a finite operational area, they can be extended to RGGs where the density of the node locations is arbitrary but has bounded support. The asymptotic behaviour here is similar to that of a graph with uniform distribution of nodes [17].

If the region of deployment is large, it makes sense to consider distributions with unbounded supports. As in other applications, this would offer us a wide variety of nice distributions which can be used to answer many interesting questions regarding the RGGs. Most interesting results for such densities depend on the tail behavior of the underlying distribution.

In this paper we primarily consider RGGs where the distribution of the node locations are i.i.d. exponential. The motivation is from random wireless sensor networks. Consider the deployment of intrusion detection sensors along a border. The cost of the sensors is expected to be significantly less than the cost of a ‘regular’ deployment. We remark here that a class of such relatively inexpensive devices called *smart dust* [18] are actually available! Hence, it is not unreasonable to expect that the sensors will be deployed by a random dispersion onto the border line. If the point from where they are dispersed is treated as the origin, it is reasonable to expect that the distribution of the sensor nodes will be dense near the origin and sparse away from it. Thus it is important to consider non-uniform distribution of the nodes. Further, analysis of networks with a finite number of nodes would also be very useful.

We remark here that the asymptotic results that are governed by the clustering of the nodes near the mode, e.g., maximum vertex degree, are obtained as in the case of RGGs with finite support [17]. In contrast, characteristics such as the largest nearest-neighbor distance, connectivity distance, minimum vertex degree etc. for densities with unbounded support are dependent on the tail-behavior and connectivity distances for normally distributed nodes are obtained in [17].

1.2 Summary of Results and Outline of Paper

Consider the exponential RGG, $G_n(\lambda, r)$ with node locations $\{X_1, \dots, X_i, \dots, X_n\}$. Let $X_{(i)}$, denote the distance of the i^{th} node from the origin or the i^{th} order statistics of the random sample $\{X_i\}$. Let $X_0 = 0$ and define $Y_i := X_{(i+1)} - X_{(i)}$ $i = 0, 1, \dots, (n-1)$. The following is a key result that we will use quite often in the remainder of this paper. From [4],

we have the following lemma.

Lemma 1. $Y_1, Y_2, \dots, Y_{n-2}, Y_{n-1}$ are independent exponential random and the means are $((n-1)\lambda)^{-1}, ((n-2)\lambda)^{-1}, \dots, (2\lambda)^{-1}, (\lambda)^{-1}$ respectively.

The lemma follows from the fact that the minimum of m i.i.d. exponentials of mean $1/\lambda$ is an exponential of mean $(m\lambda)^{-1}$ and from the memoryless property of the exponential distribution.

The rest of the paper is organized as follows. In Section 2, we derive the exact expression for the probability of connectivity P_n^c of the one dimensional exponential RGG with n nodes. In Theorem 1 we show that $P_n^c \rightarrow P_c$ as $n \rightarrow \infty$, where $0 < P_c < 1$. This limit and all other asymptotics hold under the condition that λr is fixed or converges to a constant. This is in contrast to the limiting results for the uniform and the normal case where the limiting results under the condition that $r_n \rightarrow 0$ (see [17]). In Section 3, we first give a recursive formula for the distribution of the number of components for finite n . In Theorem 2 we show that this distribution converges as $n \rightarrow \infty$ and in Theorem 3 we obtain limiting distribution for the number of components of size m . Section 4 provides a recursive formula for computing the distribution of the number of redundant nodes, nodes that can be removed without changing the connectivity of the network. In Section 5 we characterize the degree of a node by obtaining the asymptotic expectation of the degree in Theorem 4. Section 6 deals with the span and the uncovered part of the network. In Theorem 5, we show that the span of the network converges to ∞ with probability 1. However, the total number of holes (gaps between ordered nodes of length greater than r) and the total length of the holes converge in distribution. An interesting upshot of this result is that though the span of the network diverges, the probability of connectivity converges to a non-zero constant. Thus we can achieve (by taking n large) an arbitrarily large coverage with high probability, without diminishing the probability of connectivity. Theorem 6 derives the asymptotic distribution of the span of the network.

In Section 7, we derive strong law results for connectivity and largest nearest neighbor distances in Theorem 7. Finally, in Section 8 we consider RGGs where the node locations are drawn from a truncated exponential distribution, i.e., the exponential restricted to $(0, T)$. show that the asymptotic results for the truncated exponential RGG can be derived using properties of the exponential RGG $G_n(\lambda, r)$. We first define monotone properties and the strong and weak thresholds for the cutoff distance r for monotone properties. In Theorem 8 we show the equivalence of strong and weak thresholds for monotone properties in a truncated exponential RGG and an RGG constructed by considering the first n nodes of an exponential RGG. Using this, in Theorem 9 we obtain the cutoff thresholds for the RGG to be connected. Theorem 10 obtains the strong law for the connectivity and largest nearest neighbor distances.

We remark here that many of the results that we derive for the one dimensional exponential network can also be extended to the case of the nodes being distributed according to the double exponential distribution which is just the exponential density defined on the entire

real line. It has the density $\frac{\lambda}{2}e^{-\lambda|x|}$ for $-\infty < x < \infty$. We will derive only the probability of connectivity for the double exponential case.

2 Connectivity Properties

Let P_n^c denote the probability that a network of n nodes each with a transmission range r is connected. For the network to be connected we must have $Y_i = X_{(i+1)} - X_{(i)} \leq r$, $\forall i = 1, 2, \dots, (n-1)$. From Lemma 1, the following is straightforward.

Property 1. P_n^c is given by

$$P_n^c = \prod_{i=1}^{n-1} \Pr(Y_i \leq r) = \prod_{i=1}^{n-1} (1 - e^{-(n-i)\lambda r}) = \prod_{i=1}^{n-1} (1 - e^{-i\lambda r}). \quad (1)$$

We now derive the probability that a network constructed using the double exponential distribution is connected. We condition on the event that of the n nodes, k nodes are in $(0, \infty)$ and $n-k$ are in $(-\infty, 0)$. Label the positive observations as $U_i, i = 1, \dots, k$, and the absolute values of the negative observations as $V_i, i = 1, \dots, (n-k)$. Then the U_i and V_i are independent exponential variables with mean $1/\lambda$. If the network of U values is connected and the network of V values is connected and the distance between the $U_{(1)}$ and $-V_{(1)}$ is less than r , then the network will be connected. Note that from Lemma 1, it follows that $U_{(1)}$ and $V_{(1)}$ are independent of whether the networks on the positive and negative halves are connected or not. Thus, the probability that the network is connected, $P_n^c(D)$, will be

$$P_n^c(D) = \sum_{k=1}^{n-1} \binom{n}{k} (1/2)^n \Pr(U_{(1)} + V_{(1)} \leq r \mid X_{(k)} < 0, X_{(k+1)} > 0) P_k^c P_{n-k}^c + \frac{P_n^c}{2^{n-1}}. \quad (2)$$

The densities of $U_{(1)}$ and $V_{(1)}$ conditioned on the event $\{X_{(k)} < 0, X_{(k+1)} > 0\}$ will be

$$\begin{aligned} f_{U_{(1)}}(u) &= k\lambda e^{-k\lambda u} & 0 < u < \infty, \\ f_{V_{(1)}}(v) &= (n-k)\lambda e^{-(n-k)\lambda v} & -\infty < v < 0. \end{aligned}$$

The density of $(U_{(1)} + V_{(1)})$, $g_{U_{(1)}+V_{(1)}}(z)$, and hence the probability that $U_{(1)}$ and $V_{(1)}$ are connected, is now straightforward;

$$g_{U_{(1)}+V_{(1)}}(z) = \begin{cases} \frac{k(n-k)\lambda}{n-2k} (e^{-k\lambda z} - e^{-(n-k)\lambda z}) & \text{if } 2k \neq n \\ (k\lambda)^2 z e^{-k\lambda z} & \text{if } 2k = n, \end{cases}$$

and

$$\Pr(U_{(1)} + V_{(1)} \leq r) = \begin{cases} 1 + \frac{1}{n-2k} (ke^{-(n-k)\lambda r} - (n-k)e^{-k\lambda r}) & \text{if } 2k \neq n \\ 1 - e^{-k\lambda r} (1 + k\lambda r) & \text{if } 2k = n. \end{cases} \quad (3)$$

Using (1) and (3) in (2) we obtain the following.

Property 2. If the X_i are i.i.d. double exponential with zero mean, then the probability that the network is connected, $P_n^c(D)$, is given by

$$P_n^c(D) = \frac{1}{2^n} \sum_{\substack{k=0 \\ k \neq n/2}}^n \binom{n}{k} P_k^c P_{n-k}^c \left(1 + \frac{1}{n-2k} (ke^{-(n-k)\lambda r} - (n-k)e^{-k\lambda r}) \right) + \frac{(P_{n/2}^c)^2}{2^n} (1 - e^{-n\lambda r/2} (1 + n\lambda r/2)). \quad (4)$$

In (4), we have defined $P_0^c = 1$. Also, the last term will be necessary only when n is even.

Theorem 1. Let P_n^c and $P_n^c(D)$ denote the probability that the exponential and double exponential random geometric graphs respectively, with n vertices, parameter λ , and cutoff r are connected. Then, for some real number P_c , $0 < P_c < 1$,

1. $\lim_{n \rightarrow \infty} P_n^c = P_c$,
2. $\lim_{n \rightarrow \infty} P_n^c(D) = (P_c)^2$

Proof. Consider the first part of the theorem. Taking logarithms on both sides of (1) we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \ln(P_n^c) &= \sum_{i=1}^{\infty} \ln(1 - e^{-i\lambda r}) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{-(e^{-i\lambda r})^j}{j} \\ &= - \sum_{j=1}^{\infty} \frac{1}{j} \sum_{i=1}^{\infty} (e^{-j\lambda r})^i = - \sum_{j=1}^{\infty} \frac{1}{j} \frac{e^{-j\lambda r}}{1 - e^{-j\lambda r}}. \end{aligned} \quad (5)$$

Applying the ratio test we see that the series converges to a finite value $\ln P_c$. Since $-\infty < \ln(P_c) := \lim_{n \rightarrow \infty} \ln(P_n^c) < 0$ we get $0 < P_c < 1$.

Now consider the second part of the theorem statement. Let L_n be the number of nodes to the left of the origin when n nodes are distributed on the real line. By the strong law of large numbers, $\frac{L_n}{n} \xrightarrow{\text{a.s.}} \frac{1}{2}$. This implies that for any $\epsilon > 0$, there exists a finite $m(\epsilon)$ such that

$$\Pr \left(\sup_{n \geq m(\epsilon)} \left| L_n - \frac{n}{2} \right| > n\epsilon \right) < \epsilon. \quad (6)$$

To make the notation below simpler, we will assume that n is odd. Let $n > m(\epsilon)$. Define

$$A_{n,k} := \left(1 + \frac{1}{n-2k} (ke^{-(n-k)\lambda r} - (n-k)e^{-k\lambda r}) \right), \quad k = 1, \dots, (n-1). \quad (7)$$

Using the preceding definition for $A_{n,k}$, we can write (4) as

$$\begin{aligned} P_n^c(D) &= \sum_{k=1}^{n-1} \binom{n}{k} \frac{1}{2^n} P_k^c P_{n-k}^c A_{n,k} + \frac{P_n^c}{2^{n-1}} \\ &= \sum_{k: |k-n/2| \leq n\epsilon}^{n-1} \binom{n}{k} \frac{1}{2^n} P_k^c P_{n-k}^c A_{n,k} + \sum_{k: |k-n/2| > n\epsilon}^{n-1} \binom{n}{k} \frac{1}{2^n} P_k^c P_{n-k}^c A_{n,k} + \frac{P_n^c}{2^{n-1}}. \end{aligned}$$

We first consider the second term in the above equation;

$$\sum_{|k-n/2| > n\epsilon}^{n-1} \binom{n}{k} \frac{1}{2^n} P_k^c P_{n-k}^c A_{n,k} \leq \sum_{|k-n/2| > n\epsilon}^{n-1} \binom{n}{k} \frac{1}{2^n} < \epsilon. \quad (8)$$

The last inequality is derived by first observing that since k is the number of nodes to the left of the origin, the summation corresponds to the probability of $\{|L_n - n/2| > n\epsilon\}$ and then applying (6). Now consider the first sum,

$$\sum_{|k-n/2| \leq n\epsilon}^{n-1} \binom{n}{k} \frac{1}{2^n} P_k^c P_{n-k}^c A_{n,k} \leq (P_{n/2-n\epsilon}^c)^2 \sum_{\substack{k=1 \\ |k-n/2| \leq n\epsilon}}^{n-1} \binom{n}{k} \frac{1}{2^n} < (P_{n/2-n\epsilon}^c)^2. \quad (9)$$

The first inequality is true because P_n^c is decreasing in n and $A_{n,k} \leq 1$ ($A_{n,k}$ is a probability). The last inequality is true since the sum is less than 1. Also, note that $\lim_{n \rightarrow \infty} P_{n/2-n\epsilon}^c = P_c$.

We can also write the following inequality.

$$\sum_{|k-n/2| \leq n\epsilon}^{n-1} \binom{n}{k} \frac{1}{2^n} P_k^c P_{n-k}^c A_{n,k} \geq (P_{n/2+n\epsilon}^c)^2 \sum_{|k-n/2| \leq n\epsilon}^{n-1} \binom{n}{k} \frac{1}{2^n} A_{n,k} \geq P_c^2 (1 - \epsilon)^2. \quad (10)$$

The first inequality is true because P_n^c is decreasing in n . To see why the second inequality is true, we first note that $\lim_{n,k \rightarrow \infty} A_{n,k} = 1$. Hence for large $|k-n/2| > n\epsilon$ and large n $A_{n,k} > (1 - \epsilon)$. Combining this observation with (6) and noting that P_n^c converges monotonically to P_c we can write the second inequality in (10). Thus, from (9) and (10), we get

$$\lim_{n \rightarrow \infty} \sum_{|k-n/2| \leq n\epsilon}^{n-1} \binom{n}{k} \frac{1}{2^n} P_k^c P_{n-k}^c A_{n,k} = (P_c)^2.$$

Combining this with (8), the second part of the theorem is proved. \square

Numerical evaluation shows that both P_n^c and $P_n^c(D)$ converge rapidly.

3 Components in the Network

A sequence of connected nodes which are followed and preceded by a disconnected node or no nodes is called a *connected component*. In this section we derive the distribution of the number of components in the network.

Let $\{\geq j\}$ denote the network comprising of the ordered nodes $X_{(j)}, \dots, X_{(n)}$. Let $\psi_n(j, k)$, $j = 1, \dots, n$, $k = 1, \dots, n - j + 1$, denote the probability that in an n -node network there are k components in $\{\geq j\}$, $k = 1, \dots, n - j + 1$. To simplify the notation let $\zeta_i(n) := \Pr(Y_i \leq r) = (1 - e^{-\lambda(n-i)r})$. The following can be easily verified;

$$\psi_n(j, n - j + 1) = \prod_{i=j}^{n-1} (1 - \zeta_i(n)), \quad \psi_n(j, 1) = \prod_{i=j}^{n-1} \zeta_i(n). \quad (11)$$

Note that k components in $\{\geq j\}$ can occur in one of two ways; k components in $\{\geq (j+1)\}$ and nodes j and $j+1$ are connected, or $(k-1)$ components in $\{\geq (j+1)\}$ and j not connected to $j+1$. This leads us to state the following.

Property 3. *The probability that there are exactly k components in the graph, $\psi_n(1, k)$, is obtained by the recursion*

$$\psi_n(j, k) = \zeta_j(n)\psi_n(j+1, k) + (1 - \zeta_j(n))\psi_n(j+1, k-1). \quad (12)$$

The initial conditions for the recursion will be given by Eqn. 11.

We next investigate the convergence in distribution of the number of components. From Property 3 we observe that as $n \rightarrow \infty$, the number of components will essentially be determined by the last few nodes. To derive the limiting distribution of the number of components, consider the last node of the first component. Let $\theta_{n,m}$ denote the probability that node m is the last node of the first component in an n -node network, $1 \leq m \leq n$.

For any fixed m , the probability that the last node of the first component is the m^{th} from the origin goes to 0 as $n \rightarrow \infty$, but for $m = n - s$ we can obtain the following.

$$\begin{aligned} \theta_s &:= \lim_{n \rightarrow \infty} \theta_{n, n-s} \\ &= \lim_{n \rightarrow \infty} \prod_{i=1}^{m-1} (1 - e^{-r\lambda(n-i)}) e^{-r\lambda(n-m)} \\ &= \lim_{n \rightarrow \infty} \frac{P_n^c e^{-r\lambda s}}{\prod_{i=1}^{s-1} (1 - e^{-ir\lambda})} \\ &= \frac{P_c e^{-r\lambda s}}{\prod_{i=1}^{s-1} (1 - e^{-ir\lambda})}, \end{aligned} \quad (13)$$

where the last equality follows from Theorem 1. As $s \rightarrow \infty$, the denominator decreases monotonically to P_c and θ_s goes to zero as $e^{-\lambda r s}$. To obtain the limiting probability of having k components in the network, conditional on the first component ending at $m = n - s$, we need $k - 1$ components for the network composed of nodes $n - s + 1, \dots, n$. The distribution of the internodal distance between the ordered nodes $n - s + 1, \dots, n$ is exponential with parameters, $s\lambda, (s-1)\lambda, \dots, \lambda$. This is the same internodal distribution obtained when s nodes are distributed by choosing their distances from the origin to be exponentially

distributed with mean $1/\lambda$. Thus we can write the following recursive expression for the limiting probability of the network having k components.

$$\psi(1, k) := \lim_{n \rightarrow \infty} \psi_n(1, k) = \sum_{s=k}^{\infty} \theta_s \psi_s(1, k-1).$$

$P_c/(\prod_{i=1}^{s-1} (1 - e^{-ir\lambda}))$ and $\psi_s(1, k)$ are both bounded sequences. Hence, the series on the right hand side above converges. We have thus proved the following result.

Theorem 2. *For fixed λr , the number of components in the graph converges in distribution, i.e., the probability mass function for the number of components in the network converges as $n \rightarrow \infty$.*

The *size* of a component is the number of nodes in that component. We now derive an expression for the distribution of the number of components of size m . In a network with n nodes, let $P_m^n(i, k)$ denote the probability that, in $\{\geq i\}$, there are k components, each of size m . We are interested in $P_m^n(1, k)$. It is clear that if $mk > n - i + 1$, $P_m^n(i, k) = 0$. Else,

$$\begin{aligned} P_m^n(n - m + 1, 0) &= 1 - \Pr(Y_{n-m+1} \leq r, \dots, Y_{n-1} \leq r), \\ P_m^n(n - m + 1, 1) &= \Pr(Y_{n-m+1} \leq r, \dots, Y_{n-1} \leq r). \end{aligned}$$

Conditioning on the location of the first $j \geq i$ such that $Y_j > r$, we obtain a recursive relation for $P_m^n(i, k)$ as

$$\begin{aligned} P_m^n(i, k) &= \sum_{j=i+1, j \neq m+i}^{n-km+1} \Pr(Y_i \leq r, \dots, Y_{j-2} \leq r, Y_{j-1} > r) P_m^n(j, k) \\ &\quad + \Pr(Y_i \leq r, \dots, Y_{i+m-2} \leq r, Y_{i+m-1} > r) P_m^n(m+i, k-1). \end{aligned} \quad (14)$$

When $m = 1$, the first factor in the second term above should be interpreted as $\Pr(Y_i > r)$. The boundary conditions for the above recursion will be given by

$$P_m^n(i, 0) = \sum_{j=i, j \neq i+m-1}^{n-m} \Pr(Y_i \leq r, \dots, Y_{j-1} \leq r, Y_j > r) P_m^n(j+1, 0),$$

and

$$P_m^n(n - km + 1, k) = P_m^n(n - (k-1)m + 1, k-1) \Pr(Y_{n-km+1} \leq r, \dots, Y_{n-(k-1)m} > r).$$

Following the same arguments as in the proof of Theorem 2, we can derive the limiting distribution of the number of size m components.

Theorem 3. *For a fixed λr , the limiting distribution of the number of size m components is given by the following equation.*

$$P_m(k) = \lim_{n \rightarrow \infty} P_m^n(1, k) = \sum_{s=mk}^{\infty} \theta_s P_m^s(1, k), \quad (15)$$

where $P_m^s(1, k)$ are as given by (14).

By taking $m = 1$ in (15), we obtain the asymptotic distribution of the number of isolated nodes in the network.

4 Completely Covered Nodes

If there are k nodes in the interval $(X_{(i)}, X_{(i)} + r)$, then $(k - 1)$ are redundant while the k^{th} one is necessary for connectivity and we will say that $k - 1$ nodes are ‘covered’ by node i . From a sensor network perspective, the first $k - 1$ nodes in the range of node i to its right may be said to be redundant. We now determine the distribution of the number of such covered or redundant nodes in the network. Let $\phi(j, k)$, $j = 1, 2, \dots, n$ and $k = 0, 1, \dots, n - j - 1$, denote the probability that there are k redundant nodes in the network after the j^{th} node, given that the n -node network is connected. The network being connected is denoted by event C . We derive a recursive formula for $\phi(j, k)$ by conditioning on the location of the last node within the range of the j -th node. Our interest is in $\phi(1, k)$, $k = 1, 2, \dots, n - 2$.

$$\phi(j, k) = \sum_{i=j+1}^{j+k+1} \Pr(X_{(i)} \leq X_{(j)} + r < X_{(i+1)} | C) \phi(i, k - i + j + 1),$$

with boundary condition

$$\phi(j, n - j - 1) = \Pr(X_{(n)} - X_{(j)} \leq r | C).$$

$\Pr(X_{(i)} \leq X_{(j)} + r < X_{(i+1)} | C)$ is obtained as follows.

$$\begin{aligned} \Pr(X_{(i)} \leq X_{(j)} + r \leq X_{(i+1)} | C) &= \Pr((X_{(i)} - X_{(j)} \leq r) \cap (X_{(i+1)} - X_{(j)} > r) | C) \\ &= \Pr((Y_j + \dots + Y_{i-1} \leq r) \cap (Y_j + \dots + Y_i > r) | C) \\ &= \Pr((Z_{j,i} \leq r) \cap (Z_{j,i} + Y_i > r) | C), \end{aligned} \quad (16)$$

where $Z_{j,i} = Y_j + \dots + Y_{i-1}$. Since $Z_{j,i}$ is the sum of $j - i + 1$ exponentials, its density, $g_{Z_{j,i}}(z)$, is given by

$$g_{Z_{j,i}}(z) = \sum_{h=j}^{i-1} \prod_{(m=j, m \neq h)}^{i-1} \frac{n-m}{h-m} \lambda_h e^{-\lambda_h z},$$

where $\lambda_h = (n - h)\lambda$ ([20], Section 5.2.4). Using this and (16), we get

$$\Pr(X_{(i)} \leq X_{(j)} + r \leq X_{(i+1)} | C) =$$

$$\frac{\sum_{h=j}^{i-1} \left(\frac{n-i}{h-i} (e^{-\lambda_h r} - e^{-\lambda_i r}) - e^{-\lambda_h} (1 - e^{-\lambda_i r}) \right) \prod_{(m=j, m \neq h)}^{i-1} \frac{n-m}{h-m}}{\prod_{m=j}^i (1 - e^{-l(n-i)r})}. \quad (17)$$

Using the initial condition that $\phi(j, k) = 0$ for $k > n - j - 1$, the $\phi(j, k)$ can be calculated in the sequence $\phi(n - 2, 1)$, $\phi(n - 3, 1)$, $\phi(n - 3, 2)$, \dots .

5 Expected Node Degree

The degree of a node is the number of nodes lying in its range. Given a node at x , let $p(x)$ denote the probability that another node is located within distance r of x . While computing the expected number of nodes of degree k , where k is a fixed integer, we ignore the contribution to the expectation from nodes lying in $[0, r]$, since as $n \rightarrow \infty$ this contribution becomes negligible. This will happen since the number of nodes that fall in $[0, r)$ will approach ∞ and thus the vertex degrees of these nodes for fixed r will tend to ∞ . Let $W_{n,k}$ be the number of nodes of degree k , $k = 0, 1, \dots$ in an n -node network.

Theorem 4. For fixed λr as $n \rightarrow \infty$, $\lim_{n \rightarrow \infty} E(W_{n,k}) = c^{-1}$, where $c = (e^{\lambda r} - e^{-\lambda r})$ and the limit is independent of k .

Proof. Let X be an exponential random variable with parameter λ . Define,

$$p(x) = \Pr(x - r \leq X \leq x + r) = ce^{-\lambda x}, \quad x \geq r.$$

We use the notation $f(n) \sim g(n)$ to indicate that $f(n)/g(n) \rightarrow 1$, as $n \rightarrow \infty$. Since the n nodes are identically distributed, $E(W_{n,k})$ will be n times the probability that any one node in the network has degree k . Condition on this node being at x . Then, the number of nodes lying in $(x - r, x + r)$ is binomial with parameters $(n - 1)$ and $p(x)$. By the remark preceding the statement of the theorem, we ignore the contribution coming from this node lying in $[0, r)$. Hence,

$$\begin{aligned} E(W_{n,k}) &\sim n \binom{n-1}{k} \int_r^\infty p(x)^k (1 - p(x))^{n-k-1} \lambda e^{-\lambda x} dx \\ &= n \binom{n-1}{k} \int_r^\infty c^k e^{-\lambda k x} (1 - ce^{-\lambda x})^{n-k-1} \lambda e^{-\lambda x} dx \\ &\sim \frac{n^{k+1}}{k!c} \int_0^{ce^{-\lambda r}} y^k (1 - y)^{n-k-1} dy \\ &= \frac{n^{k+1}}{k!c} \int_0^{1-e^{-2\lambda r}} y^k (1 - y)^{n-k-1} dy \\ &= \frac{n^{k+1}}{k!c} \left(\int_0^1 y^k (1 - y)^{n-k-1} dy - \int_{1-e^{-2\lambda r}}^1 y^k (1 - y)^{n-k-1} dy \right). \quad (18) \end{aligned}$$

We have used the fact that $\binom{n-1}{k} \sim (n-1)^k/k!$ in deriving the second relation above. Consider the second integral in the last equation above. The function $y^k(1-y)^{n-k-1}$ has a unique maximum in $[0, 1]$ at $k/(n-1)$ which tends to zero as $n \rightarrow \infty$. Further, the function is monotonically decreasing in $(k/(n-1), 1)$. Thus the second term in the last equation above is bounded by

$$\frac{n^{k+1}}{k!c} e^{-2\lambda r} (1 - e^{-2\lambda r})^k (1 - e^{-2\lambda r})^{n-k-1},$$

which goes to zero as $n \rightarrow \infty$. The first term in (18) is

$$\frac{n^{k+1}}{k!c} Be(k+1, n-k) = \frac{n^{k+1}}{k!c} \frac{\Gamma(k+1)\Gamma(n-k)}{\Gamma(n+1)} = \frac{n^{k+1}}{cn(n-1)\dots(n-k)},$$

which converges to c^{-1} , and hence $E(W_{n,k}) \rightarrow c^{-1}$ as $n \rightarrow \infty$.

6 Span and Uncovered Segments

In $G_n(\lambda, r)$, if $Y_i > r$ we can say that there is a portion between ordered nodes i and $(i+1)$ that is not ‘covered’ and that there is a hole of size $Y_i - r$. If we think of the nodes as sensors with range r deployed along a border, then an intruder passing through the hole will go undetected. Denoting the length of the hole between the nodes i and $(i+1)$ by Z_i we have

$$Z_i = \max\{Y_i - r, 0\}.$$

The total length of the holes in the network is then $H(n, r) := \sum_{i=1}^n Z_i$ and the number of holes is $NH(n, r) := \sum_{i=1}^n I_{\{Y_i > r\}}$.

Let $S_n = X_{(n)} - X_{(1)}$ be the span of the network. Since the X_i are exponentially distributed, as $n \rightarrow \infty$, $S_n \rightarrow \infty$ almost surely. However, $H(n, r)$, the total length of the holes and $NH(n, r)$, the total number of holes in the network converge to a proper random variable in distribution.

Theorem 5. *As $n \rightarrow \infty$, $H(n, r)$ and $NH(n, r)$ converge in distribution to random variables with finite mean and variance.*

Proof. First, consider the mean and variance of $H(n, r)$ as $n \rightarrow \infty$. The density of Z_i is a shifted exponential for $z > 0$ with a point mass at 0. Thus the density of Z_i , $f_{Z_i}(z)$, can be written as

$$f_{Z_i}(z) = (1 - e^{-(n-i)\lambda r}) \delta(z) + (n-i)\lambda e^{-(n-i)\lambda(z+r)}.$$

where $\delta(z)$ is the Dirac-delta function. The mean and variance of Z_i can be shown to be given by

$$\begin{aligned} E(Z_i) &= \frac{e^{-(n-i)\lambda r}}{(n-i)\lambda}, \\ \text{Var}(Z_i) &= \frac{e^{-(n-i)\lambda r}(1 - e^{-(n-i)\lambda r})}{((n-i)\lambda)^2}. \end{aligned}$$

Since Y_1, \dots, Y_{n-1} are independent, so are the random variables Z_1, \dots, Z_{n-1} . The mean

and variance of $H(n, r)$ are then given by

$$\mathbb{E}(H(n, r)) = \sum_{k=1}^{n-1} \frac{e^{-(n-k)\lambda r}}{(n-k)\lambda} = \sum_{k=1}^{n-1} \frac{e^{-k\lambda r}}{k\lambda}, \quad (19)$$

$$\text{Var}(H(n, r)) = \sum_{k=1}^{n-1} \frac{e^{-(n-k)\lambda r} (1 - e^{-(n-k)r\lambda})}{((n-k)\lambda)^2} = \sum_{k=1}^{n-1} \frac{e^{-k\lambda r} (1 - e^{-k\lambda r})}{(k\lambda)^2}. \quad (20)$$

Applying the ratio test to the series in (19) and (20), we see that $\mathbb{E}(H(n, r))$ and $\text{Var}(H(n, r))$ converge as $n \rightarrow \infty$. Observe that since the variance of $H(n, r)$ converges to a finite limit, the usual central limit theorem will not be applicable.

To show convergence in distribution, we must show that the sequence of random variables $\{H(n, r)\}$ is tight (relatively compact) and that the Laplace transform of $H(n, r)$ converges (see Lemma 2, pp. 323 in [21]). Tightness means that the probability of the $H(n, r)$ lying outside a compact set can be made arbitrarily small. Tightness also implies that any subsequence $H(n_k, r)$ of $H(n, r)$ will contain a subsequence that converges in distribution. We need to show tightness because of the absence of a nice closed form expression for the characteristic function of $H(n, r)$. Convergence of the Laplace transform implies uniqueness of these limits thereby implying convergence in distribution.

To show tightness we need to show that for any $\epsilon > 0$, there exists a $K > 0$ such that $\sup_{n \geq 1} \Pr(H(n, r) > K) < \epsilon$. $H(n, r)$ are nonnegative random variables and we can use Markov inequality to write, for any $K > 0$,

$$\Pr(H(n, r) > K) \leq \frac{\mathbb{E}(H(n, r))}{K}.$$

Since $\mathbb{E}(H(n, r))$ converges and is finite, for any ϵ , a sufficiently large K can be found such that $\mathbb{E}(H(n, r)) / K < \epsilon$. Thus the random variables $H(n, r)$ are tight.

To complete the proof of convergence in distribution of $H(n, r)$, we have to show that the Laplace transform $L_n(\theta)$ of $H(n, r)$, converges in some neighborhood of zero.

$$\begin{aligned} L_n(\theta) &:= \mathbb{E}(e^{\theta H(n, r)}) = \mathbb{E}\left(e^{\theta \sum_{i=1}^{n-1} Z_i}\right) = \prod_{i=1}^{n-1} \mathbb{E}(e^{\theta Z_i}) \\ &= \prod_{i=1}^{n-1} \left(1 + \frac{\theta e^{-(n-i)\lambda r}}{(n-i)\lambda - \theta}\right) \quad \theta < \lambda. \end{aligned}$$

Taking logarithms on both sides, we get

$$\begin{aligned}
\ln(L_n(\theta)) &= \sum_{i=1}^{n-1} \ln \left(1 + \frac{\theta e^{-(n-i)\lambda r}}{((n-i)\lambda - \theta)} \right) \\
&= \sum_{i=1}^{n-1} \ln \left(1 + \frac{\theta e^{-ir\lambda}}{(\lambda i - \theta)} \right) \\
&\leq \sum_{i=1}^{n-1} \frac{\theta e^{-i\lambda r}}{(\lambda i - \theta)}.
\end{aligned}$$

The last inequality above is obtained from the inequality $\ln(1+x) \leq x$. Observe that $\sum_{i=1}^{\infty} \frac{\theta e^{-i\lambda r}}{(\lambda i - \theta)}$ converges by ratio test. This proves the convergence of $L_n(\theta)$ and hence the second part of the theorem on the convergence of $H(n, r)$ in distribution.

We now consider convergence in distribution of the number of holes. The mean and variance of $NH(n, r)$ are given by

$$\begin{aligned}
\mathbb{E}(NH(n, r)) &= \sum_{i=1}^{n-1} e^{-(n-i)\lambda r} = \frac{e^{-r\lambda}(1 - e^{-(n-1)r\lambda})}{1 - e^{-r\lambda}}, \\
\text{Var}(NH(n, r)) &= \sum_{j=1}^{n-1} e^{-\lambda_j r} (1 - e^{-\lambda_j r}).
\end{aligned}$$

Application of the ratio test shows that both the above series converge. Tightness of $NH(n, r)$ follows by the same argument as that used to show the tightness of $H(n, r)$. The Laplace transform of $NH(n, r)$, $J_n(\theta)$, is given by

$$J_n(\theta) = \prod_{i=1}^{n-1} (1 - e^{-i\lambda r}(1 - e^{\theta})). \quad (21)$$

Convergence of $J_n(\theta)$ can be shown as for $L_n(\theta)$. Thus $NH(n, r)$ converges in distribution as $n \rightarrow \infty$.

This completes the proof of the Theorem. \square

Theorem 5 implies that for large n , we can increase the span of the network over any length with a certain high probability, by adding more nodes without a corresponding increase in the length of the holes or the number of holes.

Remark 1. *Since the number of components is just one more than the number of holes, the convergence in distribution of the number of holes follows from Theorem 2. Thus this is an alternate proof for Theorem 2. The limit of $J_n(\theta)$ can be used to obtain the asymptotic moments for the number of components.*

The asymptotic distribution of the span is also known. From Examples 3.3 and 3.5 of [5], we have that $\lambda^{-1}X_{(1)} \log(n/(n-1))$ converges in distribution to a Weibull distribution and

$\lambda X_{(n)} - \log(n)$ converges in distribution to a Gumbel distribution. This allows us to state the following result for the asymptotic distribution of the span.

Theorem 6. $\lambda(X_{(n)} - X_{(1)}) - \log(n)$ converges in distribution to a Gumbel distribution.

Thus, the $100(1 - \alpha)\%$ confidence interval for the span based on the asymptotic distribution will be of the form $\lambda^{-1}(\log(n) \pm c(\alpha))$ where $c(\alpha)$, is independent of n .

7 Strong Law Results

In this section we derive almost sure convergence results for the connectivity and the largest nearest neighbor distances and a limiting result for the almost surely connected part of the exponential random geometric graph.

Define c_n and d_n the connectivity and largest nearest neighbor distances respectively as

$$c_n = \inf\{r > 0 : G_n(\lambda, r) \text{ is connected}\}, \quad (22)$$

$$d_n := \max_{1 \leq i \leq n} \min_{1 \leq j \leq n, j \neq i} \{|X_i - X_j|\} \quad (23)$$

Theorem 7. For fixed $\lambda > 0$,

1.

$$\limsup_{n \rightarrow \infty} \frac{\lambda c_n}{\ln(n)} = \limsup_{n \rightarrow \infty} \frac{\lambda d_n}{\ln(n)} = 1, \quad \text{almost surely.} \quad (24)$$

2.

$$\liminf_{n \rightarrow \infty} \frac{\lambda \ln(n) c_n}{c} \geq 1, \quad \liminf_{n \rightarrow \infty} \frac{\lambda \ln(n) d_n}{c} \geq 1, \quad \text{almost surely.} \quad (25)$$

where $c = \sum_{j=1}^{\infty} j^{-2}$.

3. Let r be fixed, $k_n = \lfloor n(1 - a \ln(n)/n) \rfloor$ where $\lfloor \cdot \rfloor$ denotes the integer part and $a > (\lambda r)^{-1}$. Let $G_n(k_n, \lambda, r)$ denote the graph $G_n(\lambda, r)$ restricted to the first k_n ordered points. Then,

$$\Pr(G_n(k_n, \lambda, r) \text{ is disconnected infinitely often}) = 0. \quad (26)$$

Proof.

$$\Pr(c_n \geq y) = \Pr(\cup_{i=1}^{n-1} \{Y_i \geq y\}) \leq \sum_{i=1}^{n-1} e^{-\lambda i y} = e^{-\lambda y} \frac{1 - e^{-(n-1)\lambda y}}{1 - e^{-\lambda y}}.$$

Taking $y = (1 + \epsilon) \log(n)/\lambda$, and applying the ratio test, we see that

$$\sum_{n=2}^{\infty} \Pr(\lambda c_n \geq (1 + \epsilon) \log(n)) < \infty.$$

By the Borel-Cantelli lemma, $\Pr(\lambda c_n \geq (1 + \epsilon) \log(n) \text{ i.o. }) = 0$. Since $\epsilon > 0$ is arbitrary, we conclude that $\limsup(\lambda c_n / \log(n)) \leq 1$ a.s.

To show that the \limsup is exactly equal to one, consider the record values denoted by R_n defined as follows: Let $N(1) = 1$. For $n \geq 2$, define $N(n) = \inf\{k > N(n-1) : X_k > X_{N(n-1)}\}$. Define $R_n = X_{N(n)}$. Since the exponential density has unbounded support, there will be a.s. infinitely many record values. Consider the sequence $Z_n = R_n - R_{n-1}$, $n \geq 2$. By the memoryless property of the exponential, Z_n is a sequence of independent exponential random variables with mean λ^{-1} . Since, for any $\epsilon > 0$,

$$\sum_{n=2}^{\infty} \Pr(\lambda Z_n > (1 - \epsilon) \log(n)) = \sum_{n=2}^{\infty} n^{-(1-\epsilon)} = \infty,$$

it follows from the Borel-Cantelli Lemma that $\limsup \lambda Z_n / \log(n) = 1$ a.s. The above result implies that $\limsup(\lambda d_n / \log(n)) \geq 1$ a.s. by considering the sequence of graphs $G_{N(k)}(\lambda, r)$. Part 1 of the theorem now follows because $d_n \leq c_n$.

To prove part 2 for c_n , we consider the asymptotic behavior of the probability that $G_n(\lambda, r_n)$ is connected for the sequence of cutoff distances $r_n = c / (\lambda(1 + \epsilon) \ln(n))$, where c is as defined in the theorem statement.

$$P_c^n = \Pr(G_n(\lambda, r_n) \text{ is connected}) = \prod_{i=1}^{n-1} (1 - \exp(-\lambda i r_n)).$$

Taking logarithms and expanding the logarithm, we get

$$\ln(P_c^n) = - \sum_{i=1}^{n-1} \sum_{j=1}^{\infty} \frac{e^{-\lambda i j r_n}}{j} = - \sum_{j=1}^{\infty} \frac{e^{-\lambda j r_n} (1 - e^{-\lambda j (n-1) r_n})}{j(1 - e^{-\lambda j r_n})}.$$

Since $r_n \rightarrow 0$, and $n r_n \rightarrow \infty$, we have $e^{-\lambda j r_n} \rightarrow 1$, $1 - e^{-\lambda j (n-1) r_n} \rightarrow 1$, and $1 - e^{-\lambda j r_n} \sim \lambda j r_n$. Hence,

$$\ln(P_c^n) \sim - \frac{1}{\lambda r_n} \sum_{j=1}^{\infty} j^{-2}.$$

Plugging in the expression for r_n we get $P_c^n \sim n^{-(1+\epsilon)}$, which is summable. The result for c_n in Part 2 now follows from the Borel-Cantelli lemma. To prove part 2 for d_n let $y_n = \frac{c}{\lambda(1+\epsilon) \log n}$, and consider,

$$\begin{aligned} \Pr(d_n \leq y_n) &= \Pr\left(\bigcap_{i=2}^{n-1} ((Y_{i-1} \leq y_n) \cup (Y_i \leq y_n)) \cap (Y_1 \leq y_n) \cap (Y_{n-1} \leq y_n)\right) \\ &\leq \Pr\left(\bigcap_{i=1}^{\lfloor n/2 \rfloor} ((Y_{2i-1} \leq y_n) \cup (Y_{2i} \leq y_n))\right) \\ &\leq \prod_{i=1}^{\lfloor n/2 \rfloor} (1 - e^{-\lambda(2n-4i-1)y_n}). \end{aligned}$$

Take logarithms on both sides and using the Taylor expansion, we get,

$$\begin{aligned}
\ln(\Pr(d_n \leq y_n)) &= - \sum_{j=0}^{\infty} \sum_{i=1}^{\lfloor n/2 \rfloor} \frac{e^{-\lambda(2n-4i+1)jy_n}}{j} \\
&= - \sum_{j=0}^{\infty} \frac{e^{-\lambda jy_n(2n+1)}}{j} \sum_{i=1}^{\lfloor n/2 \rfloor} (e^{4\lambda jy_n})^i \\
&= - \sum_{j=0}^{\infty} \frac{1}{j} e^{-\lambda jy_n(2n-3)} \frac{1 - e^{4\lambda jy_n \lfloor n/2 \rfloor}}{1 - e^{4\lambda jy_n}} \\
&= - \sum_{j=0}^{\infty} \frac{1}{j} \frac{e^{-\lambda jy_n(2n-3)} - e^{4\lambda jy_n(2n-3-4\lfloor n/2 \rfloor)}}{1 - e^{4\lambda jy_n}} \\
&\sim \frac{-1}{4\lambda y_n} \sum_{j=0}^{\infty} \frac{1}{j^2},
\end{aligned}$$

where the last approximation follows since $y_n \rightarrow 0$ and $ny_n \rightarrow \infty$ which implies that $\exp(-\lambda jy_n(2n-3)) \rightarrow 0$, $\exp(4\lambda jy_n(2n-3-4\lfloor n/2 \rfloor)) \sim \exp(-12\lambda jy_n) \rightarrow 1$, and $1 - \exp(4\lambda jy_n) \sim -4\lambda jy_n$. Substituting for y_n we get,

$$P[d_n \leq \frac{c}{\lambda(1+\epsilon)\log n}] \sim \frac{1}{n^{1+\epsilon}}, \quad (27)$$

which is summable. Part 2 of the theorem for d_n now follows from the Borel-Cantelli Lemma.

To prove part 3, consider

$$\begin{aligned}
\Pr(G_n(k_n, \lambda, r) \text{ is not connected}) &\leq \sum_{i=1}^{k_n-1} e^{-\lambda r(n-i)} \\
&= \frac{e^{\lambda r}}{e^{\lambda r} - 1} (e^{-\lambda r(n-k_n)} - e^{-\lambda r n}).
\end{aligned}$$

For large n , $n - k_n \sim a \log(n)$, and hence the above probability is summable. The result now follows from the Borel-Cantelli Lemma.

8 Truncated Exponential Graph

We now consider the RGG $G_n(\lambda, r, T)$ where the nodes are distributed independently according to the density function

$$g_{\lambda, T}(x) = \frac{\lambda e^{-\lambda x}}{1 - e^{-\lambda T}}, \quad 0 \leq x \leq T,$$

and have a cutoff r . This distribution allows us to consider RGGs with finite support where the distribution of the nodes is not uniform. We derive asymptotic results for the connectivity and largest nearest neighbor distances for this graph. [17] derives similar results for dimensions $d \geq 2$ and for general densities having bounded support. [2, 3] show such strong laws for the uniform RGG for $d \geq 1$. [13] obtains strong law results for the one dimensional uniform RGG using the graph with independent exponential spacings of [15].

In deriving our results, note that unlike in the exponential RGG, the spacings in $G_n(\lambda, r, T)$ are not independent. Our proof technique is as follows. We show that the graph $G_n(\lambda, r, T)$ has the same asymptotic behavior as that of a graph G_n^* which is constructed by considering the first n nodes of an exponential RGG on N vertices. Here $N = N(n) := \lfloor n/p \rfloor$ and $p = 1 - \exp(-\lambda T)$. Spacings in the graph G_n^* are independent and hence it is possible to derive results easily for this graph. This technique allows us to think about $G_n(\lambda, r, T)$ in terms of the graph G_n^* whose properties can be more easily visualized. This is similar to the approach of [15] for the uniform RGG.

Let X_1, X_2, \dots be a sequence of independent random variables with density $g_{\lambda, T}$. The vertex set of $G_n(\lambda, r, T)$ is $V_n = \{X_1, \dots, X_n\}$. Let $N(n)$ be as defined above and let Z_1, Z_2, \dots , be a sequence of exponential random variables with mean λ^{-1} . Let $Z_{1,N}, \dots, Z_{N,N}$ denote the ordered values of the first $N(n)$ random variables Z_1, \dots, Z_N . Define the graph $G_n^*(\lambda, r)$ to be the RGG with cutoff r and vertex set $V_n^* = \{Z_{1,N}, \dots, Z_{n,N}\}$. We denote by $G_n^*(\lambda, r, t)$ the graph with vertex set V_n^* conditioned on $Z_{n+1,N} = t$. It is easy to see that the conditional density of first n ordered observations $Z_{1,N}, \dots, Z_{n,N}$ given $Z_{n+1,N} = t$ is given by (see [19], pp. 175–176),

$$f_{Z_{1,N}, \dots, Z_{n,N} | Z_{n+1,N}}(z_1, \dots, z_n | t) = \frac{n! \lambda^n}{(1 - e^{-\lambda t})^n} e^{-\lambda \sum_{i=1}^n z_i}, \quad (28)$$

for $0 < z_1 < \dots < z_n < t$. The key observation is that the above function is also the joint density function of n i.i.d. ordered observations from $g_{\lambda, t}$. Further, we have the following lemma which states that $Z_{n+1,N}$ is close to T with large probability as $n \rightarrow \infty$. Subsequent to this lemma, we show that the graphs G_n and G_n^* have the same asymptotic behavior.

Lemma 2. $Z_{n+1,N} \rightarrow T$ in probability as $n \rightarrow \infty$.

Proof. We show that the mean and variance of $Z_{n+1,N}$ converge to T and 0 respectively. The result then follows from Chebyshev's inequality.

$$\mathbb{E}(Z_{n+1,N}) = \sum_{i=0}^n \frac{1}{(N-i)\lambda} = \frac{1}{\lambda} \sum_{i=N-n}^N \frac{1}{i}.$$

Hence,

$$\int_{N-n}^{N+1} \frac{1}{x} dx \leq \lambda \mathbb{E}(Z_{n+1,N}) \leq \int_{N-n-1}^N \frac{1}{x} dx.$$

Both the integrals above converge to λT as $n \rightarrow \infty$ by the definition of N .

$$\begin{aligned}\text{Var}(Z_{n+1,N}) &= \sum_{i=0}^n \frac{1}{(N-i)^2 \lambda^2} = \frac{1}{\lambda^2} \sum_{i=N-n}^N \frac{1}{i^2} \\ &\leq \frac{1}{\lambda^2} \int_{N-n-1}^N \frac{1}{x^2} dx \rightarrow 0.\end{aligned}$$

Thus for any $\epsilon > 0$, and n sufficiently large, we have $|E(Z_{n+1,N}) - T| < \epsilon/2$. Hence

$$\begin{aligned}\Pr(|Z_{n+1,N} - T| > \epsilon) &\leq \Pr(|Z_{n+1,N} - E(Z_{n+1,N})| > \epsilon/2) \\ &\leq \frac{4\text{Var}(Z_{n+1,N})}{\epsilon^2} \rightarrow 0.\end{aligned}$$

This completes the proof the lemma. \square

We now show that the graphs G_n and G_n^* have the same thresholding behavior. To do this we need some notations.

Definition 1. If A and B are graphs such that A and B share the same vertices, and the edge set of A is a subset of the edge set of B , we will write $A \leq B$. Let Θ be a property of a random geometric graphs such that if $A \leq B$ and $A \in \Theta$, then $B \in \Theta$. (Here $A \in \Theta$ is used to denote that RGG A has property Θ .) Then Θ is called an “upwards-closed” property. If $B \in \Theta$ implies $A \in \Theta$, then Θ is said to be a “downwards-closed” property.

Fix an upwards-closed property Θ . For any two functions $\delta, \gamma : Z^+ \rightarrow \mathbb{R}^+$, we write $\delta \ll \gamma$ (resp. $\delta \gg \gamma$) if $\delta(n)/\gamma(n) \rightarrow 0$, (resp. $\gamma(n)/\delta(n) \rightarrow 0$) as $n \rightarrow \infty$. In what follows we will write δ for $\delta(n)$. Let $G_n(r)$, be any random geometric graph on n vertices with cutoff r .

Definition 2. A function $\delta_\Theta : Z^+ \rightarrow \mathbb{R}^+$ is a weak threshold function for Θ if the following is true for every function $\delta : Z^+ \rightarrow \mathbb{R}^+$,

- if $\delta(n) \ll \delta_\Theta(n)$, then $\Pr(G_n(\delta) \in \Theta) = o(1)$, and
- if $\delta(n) \gg \delta_\Theta(n)$ then $\Pr(G_n(\delta) \in \Theta) = 1 - o(1)$.

A function $\delta_\Theta : Z^+ \rightarrow \mathbb{R}^+$ is a strong threshold function for Θ if the following is true for every fixed $\epsilon > 0$,

- if $\Pr(G_n((1 - \epsilon)\delta_\Theta) \in \Theta) = o(1)$, and
- if $\Pr(G_n((1 + \epsilon)\delta_\Theta) \in \Theta) = 1 - o(1)$.

Before proceeding further, we show the following two monotonicity properties that will be used subsequently.

Lemma 3. Let Θ be any upwards-closed property. Then,

1. For any $0 < T_1 < T_2$, $\Pr(G_n(\lambda, \delta, T_1) \in \Theta) \geq \Pr(G_n(\lambda, \delta, T_2) \in \Theta)$.
2. For any $0 < \lambda_1 < \lambda_2$, $\Pr(G_n^*(\lambda_1, \delta) \in \Theta) \leq \Pr(G_n^*(\lambda_2, \delta) \in \Theta)$.
3. For any $c > 0$, $\Pr(G_n(\lambda, \delta, T) \in \Theta) = \Pr(G_n(c^{-1}\lambda, c\delta, cT) \in \Theta)$.

Proof. Let $U_{(1)}, U_{(2)}, \dots, U_{(n)}$ be n ordered uniform random variables on $(0, 1)$. The ordered vertex sets of the graphs $G_n(\lambda, \delta, T_1)$, $G_n(\lambda, \delta, T_2)$, $G_n^*(\lambda_1, \delta)$ and $G_n(\lambda_2, \delta)$ may be defined using the ordered uniform variables as follows: $V_1 = \{-\frac{1}{\lambda} \ln(1 - U_{(i)}(1 - e^{-\lambda T_1}))\}_{i=1}^n$, $V_2 = \{-\frac{1}{\lambda} \ln(1 - U_{(i)}(1 - e^{-\lambda T_2}))\}_{i=1}^n$, $V_3 = \{-\frac{1}{\lambda_1} \ln(1 - U_{(i)})\}_{i=1}^n$, and $V_4 = \{-\frac{1}{\lambda_2} \ln(1 - U_{(i)})\}_{i=1}^n$. If we denote the respective edge sets by E_i , $i = 1, \dots, 4$, then it is easy to see that $E_2 \subset E_1$ and $E_3 \subset E_4$. The result for parts 1 and 2 now follows from the definition of an upwards-closed property. To prove part 3, observe that Θ being an upwards-closed property depends only on the existence of edges between certain pairs of nodes in a given configuration of vertices (Z_1, \dots, Z_n) of the graph $G_n(\lambda, \delta, T)$. Let $A \subset \mathbb{R}^n$ be such that Θ holds whenever $(Z_1, \dots, Z_n) \in A$ at cutoff δ , then clearly, it holds at cutoff $c\delta$ if $(Z_1, \dots, Z_n) \in cA$. The joint density of n independent truncated exponential random variables on $[0, T]$ is given by (28), with t replaced by T . Hence,

$$\Pr(G_n(\lambda, \delta, T) \in \Theta) = \int_{\{(z_1, \dots, z_n) \in A\}} \frac{n! \lambda^n}{(1 - e^{-\lambda T})^n} e^{-\lambda \sum_{i=1}^n z_i} dz_1 \dots dz_n. \quad (29)$$

From the above remarks on upwards-closed property and (29), we have

$$\begin{aligned} \Pr(G_n(c^{-1}\lambda, c\delta, cT) \in \Theta) &= \int_{\{(z_1, \dots, z_n) \in cA\}} \frac{n! c^{-n} \lambda^n}{(1 - e^{-c^{-1}\lambda cT})^n} e^{-c^{-1}\lambda \sum_{i=1}^n z_i} dz_1 \dots dz_n \\ &= \int_{\{c^{-1}(z_1, \dots, z_n) \in A\}} \frac{n! c^{-n} \lambda^n}{(1 - e^{-\lambda T})^n} e^{-c^{-1}\lambda \sum_{i=1}^n z_i} dz_1 \dots dz_n \end{aligned}$$

Change variables $cu_i = z_i$, $i = 1, \dots, n$.

$$\begin{aligned} &= \int_{\{(u_1, \dots, u_n) \in A\}} \frac{n! \lambda^n}{(1 - e^{-\lambda T})^n} e^{-\lambda \sum_{i=1}^n u_i} du_1 \dots du_n \\ &= \Pr(G_n(\lambda, \delta, T) \in \Theta), \end{aligned}$$

which proves part 3.

Lemma 4. Let $\delta : Z^+ \rightarrow \mathbb{R}^+$, $T > 0$ and $\alpha \in (0, 1)$. Let $G_n(\lambda, \delta, T)$ and $G_n^*(\lambda, \delta)$ be the random geometric graphs defined above. Then for all n sufficiently large the following hold.

1. If $\Pr(G_n^*(\lambda, \delta) \in \Theta) \leq \alpha$, then $\Pr(G_n(\lambda, (1 - \epsilon)\delta, T) \in \Theta) \leq \frac{\alpha}{1 - \alpha}$.
2. If $\Pr(G_n(\lambda, \delta, T) \in \Theta) \leq \alpha$, then $\Pr(G_n^*(\lambda, (1 - \epsilon)\delta) \in \Theta) \leq 2\alpha$.
3. If $\Pr(G_n^*(\lambda, \delta) \in \Theta) \geq 1 - \alpha$, then $\Pr(G_n(\lambda, (1 + \frac{\epsilon}{1 - \epsilon})\delta, T) \in \Theta) \geq \frac{1 - 2\alpha}{1 - \alpha}$.

4. If $\Pr(G_n(\lambda, \delta, T) \in \Theta) \geq 1 - \alpha$, then $\Pr(G_n^*(\lambda, (1 + \epsilon)\delta) \in \Theta) \geq 1 - 2\alpha$.

Proof. Let $Z_{(n+1)} = Z_{n+1,N}$, be the random variables defined prior to Lemma 2. For any $\epsilon, \alpha \geq 0$, from Lemma 2, there exists a $M \geq 0$ such that, for all $n \geq M$,

$$\Pr(|Z_{(n+1)} - T| \leq \epsilon) \geq 1 - \alpha \quad (30)$$

For the sake of simplicity we will take $T = 1$ in this proof and write $G_n(\lambda, \delta)$ for $G_n(\lambda, \delta, 1)$.

$$\begin{aligned} & \Pr(G_n^*((1 - \epsilon)\lambda, \delta) \in \Theta) \\ &= \int_0^\infty \Pr(G_n^*((1 - \epsilon)\lambda, \delta) \in \Theta \mid Z_{(n+1)} = z) f_{Z_{(n+1)}}(z) dz \\ &\geq \int_{1-\epsilon}^{1+\epsilon} \Pr(G_n^*((1 - \epsilon)\lambda, \delta) \in \Theta \mid Z_{(n+1)} = z) f_{Z_{(n+1)}}(z) dz \\ &= \int_{1-\epsilon}^{1+\epsilon} \Pr(G_n((1 - \epsilon)\lambda, \delta, z) \in \Theta) f_{Z_{(n+1)}}(z) dz \\ &\geq \Pr(G_n((1 - \epsilon)\lambda, \delta, (1 + \epsilon)) \in \Theta) \int_{1-\epsilon}^{1+\epsilon} f_{Z_{(n+1)}}(z) dz. \end{aligned}$$

In deriving the second inequality above we have used Lemma 3(1). Using (30), we get

$$\Pr(G_n^*((1 - \epsilon)\lambda, \delta) \in \Theta) \geq (1 - \alpha) \Pr(G_n((1 - \epsilon)\lambda, \delta, (1 + \epsilon)) \in \Theta),$$

for all $n \geq M$. Since $\Pr(G_n^*(\lambda, \delta) \in \Theta) \leq \alpha$, it follows from Lemma 3(2) that

$$\begin{aligned} \Pr(G_n^*((1 - \epsilon)\lambda, \delta) \in \Theta) &\leq \alpha, \\ \Pr(G_n((1 - \epsilon)\lambda, \delta, (1 + \epsilon)) \in \Theta) &\leq \frac{\alpha}{1 - \alpha}. \end{aligned}$$

Hence, by Lemma 3(3),

$$\Pr(G_n(\lambda, (1 - \epsilon)\delta, (1 + \epsilon)(1 - \epsilon)) \in \Theta) \leq \frac{\alpha}{1 - \alpha}.$$

Since $(1 + \epsilon)(1 - \epsilon) \leq 1$, part 1 of the lemma follows from another application of Lemma 3(1).

To prove part 2, consider

$$\begin{aligned} & \Pr(G_n^*((1 - \epsilon)^{-1}\lambda, (1 - \epsilon)\delta) \in \Theta) \\ &= \int_0^\infty \Pr(G_n^*((1 - \epsilon)^{-1}\lambda, (1 - \epsilon)\delta) \in \Theta \mid Z_{(n+1)} = z) f_{Z_{(n+1)}}(z) dz \\ &\leq \alpha + \int_{1-\epsilon}^{1+\epsilon} \Pr(G_n^*((1 - \epsilon)^{-1}\lambda, (1 - \epsilon)\delta) \in \Theta \mid Z_{(n+1)} = z) f_{Z_{(n+1)}}(z) dz \\ &= \alpha + \int_{1-\epsilon}^{1+\epsilon} \Pr(G_n((1 - \epsilon)^{-1}\lambda, (1 - \epsilon)\delta, z) \in \Theta) f_{Z_{(n+1)}}(z) dz \\ &\leq \alpha + \Pr(G_n((1 - \epsilon)^{-1}\lambda, (1 - \epsilon)\delta, (1 - \epsilon)) \in \Theta) \int_{1-\epsilon}^{1+\epsilon} f_{Z_{(n+1)}}(z) dz, \\ &\leq \alpha + \Pr(G_n(\lambda, \delta, 1) \in \Theta), \end{aligned}$$

for all $n \geq N$. The first inequality above follows from (30). The second inequality follows from Lemma 3(1) while the last inequality follows from Lemma 3(3). From the given condition $\Pr(G_n(\lambda, \delta, 1) \in \Theta) \leq \alpha$, it follows that

$$\Pr(G_n^*((1-\epsilon)^{-1}\lambda, (1-\epsilon)\delta) \in \Theta) \leq \alpha + \alpha = 2\alpha.$$

Part 2 now follows from Lemma 3(2).

To prove part 3, proceeding as above, we get

$$\begin{aligned} & \Pr(G_n^*((1-\epsilon)^{-1}\lambda, \delta) \notin \Theta) \\ &= \int_0^\infty \Pr(G_n^*((1-\epsilon)^{-1}\lambda, \delta) \notin \Theta \mid Z_{n+1} = z) f_{Z_{n+1}}(z) dz \\ &\geq \int_{1-\epsilon}^{1+\epsilon} \Pr(G_n^*((1-\epsilon)^{-1}\lambda, \delta) \notin \Theta \mid Z_{n+1} = z) f_{Z_{n+1}}(z) dz \\ &= \int_{1-\epsilon}^{1+\epsilon} \Pr(G_n((1-\epsilon)^{-1}\lambda, \delta, z) \notin \Theta) f_{Z_{n+1}}(z) dz \\ &\geq \Pr(G_n((1-\epsilon)^{-1}\lambda, \delta, (1-\epsilon)) \notin \Theta) \int_{1-\epsilon}^{1+\epsilon} f_{Z_{n+1}}(z) dz, \\ &\geq (1-\alpha) \Pr(G_n((1-\epsilon)^{-1}\lambda, \delta, 1-\epsilon) \notin \Theta), \end{aligned}$$

for all $n \geq M$. Since the given condition $\Pr(G_n^*(\lambda, \delta) \notin \Theta) \leq \alpha$, implies $\Pr(G_n^*((1-\epsilon)^{-1}\lambda, \delta) \notin \Theta) \leq \alpha$, it follows that

$$\Pr(G_n((1-\epsilon)^{-1}\lambda, \delta, 1-\epsilon) \notin \Theta) \leq \frac{\alpha}{1-\alpha}$$

This implies that

$$\Pr\left(G_n(\lambda, (1 + \frac{\epsilon}{1-\epsilon})\delta) \in \Theta\right) \geq \frac{1-2\alpha}{1-\alpha},$$

and we have the proof for part 3.

To prove part 4 we proceed as above to get the following inequality.

$$\Pr(G_n^*(\lambda, (1+\epsilon)\delta) \notin \Theta) \leq \alpha + \Pr(G_n(\lambda, (1+\epsilon)\delta, 1+\epsilon) \notin \Theta) \leq 2\alpha.$$

This completes the proof the lemma.

Remark 2. *The above results extend to downwards-closed properties as well.*

The following theorem is now an easy corollary of the above Lemma.

Theorem 8. *The sequence of random geometric graphs $G_n(\lambda, \delta, T)$ and $G_n^*(\lambda, \delta)$ have the same weak and strong thresholds.*

Remark 3. *Proceeding as in the proof of Lemma 4, we can show that*

$$|\Pr(G_n(\lambda, \delta, T) \in \Theta) - \Pr(G_n^*(\lambda, \delta) \in \Theta)| \rightarrow 0, \quad n \rightarrow \infty.$$

This implies that all the asymptotic probabilities of $G_n(\lambda, \delta, T)$ satisfying any monotone property (upward or downward closed) can be obtained by studying the corresponding probabilities for $G_n^(\lambda, \delta)$.*

We now use Theorem 8 to derive the threshold probability for connectivity and strong law results for the connectivity and largest nearest neighbor distance for the graph G_n . We will, without any further reference to the above theorem, work with the graph G_n^* instead of G_n .

Theorem 9. *Let $p = 1 - \exp(-\lambda T)$. Then the sequence of edge distances $\delta(n) = \frac{p}{\lambda(1-p)} \frac{\ln(n)}{n}$ is a strong (and weak) threshold for connectivity for the graph $G_n(\lambda, \delta, T)$.*

Proof. Let $r_n = a\delta(n)$, where $a \geq 0$ is a constant. Note that $r_n \rightarrow 0$ while $nr_n \rightarrow \infty$ and $n \sim Np$. Let P_n^{*c} be the probability that $G_n^*(\lambda, r_n)$ is connected. Then,

$$\ln(P_n^{*c}) = \sum_{j=N-n}^N \ln(1 - e^{-\lambda r_n j}).$$

Since $jr_n \rightarrow \infty$ for all $j = N - n, \dots, N$, using $\ln(1 - x) \sim x$ as $x \rightarrow 0$, and summing the resultant geometric series, we get

$$\ln(P_n^{*c}) \sim -e^{-\lambda r_n(N-n)} \frac{1 - e^{-\lambda r_n(n+1)}}{1 - e^{-\lambda r_n}}.$$

Substituting for $r_n = a\delta(n)$ while noting that $(N - n)/n \sim (1 - p)/p$, $1 - e^{-\lambda r_n(n+1)} \rightarrow 1$ and $1 - e^{-\lambda r_n} \sim -\lambda r_n$, we obtain,

$$\ln(P_n^{*c}) \sim -\frac{n^{1-a}}{\ln(n)}.$$

Thus, for $a = 1 + \epsilon$, P_n^{*c} converges to 1 and converges to 0 for $a = 1 - \epsilon$. This shows that $\delta(n)$ is a strong threshold for connectivity for G_n . Similarly one can show that $\delta(n)$ is a weak threshold as well.

Remark 4. *Note that $\frac{n\delta(n)}{\ln(n)} = \frac{p}{\lambda(1-p)}$, where $\frac{p}{\lambda(1-p)}$ is the reciprocal of the minimum $g_{\lambda,T}(x)$. Thus the behavior of the distance required to connect the graph is determined by the minimum of the density since in the vicinity of this point vertices are more sparsely distributed. The normalization $\frac{n}{\ln(n)}$ is the same as in the case of uniform distribution of nodes.*

We now state a strong law result for the connectivity distance ($c_n(\lambda, T)$) and the largest nearest neighbor distance ($d_n(\lambda, T)$) for $G_n(\lambda, \cdot, T)$. In the following, we drop the reference to parameters λ and T when referring to $c_n(\lambda, T)$ and $d_n(\lambda, T)$.

Theorem 10. Let $\lambda, T > 0$. The connectivity and largest nearest neighbor distances of the graph $G_n(\lambda, \cdot, T)$ satisfy

1. $\lim_{n \rightarrow \infty} \frac{nc_n}{\ln(n)} = \frac{p}{\lambda(1-p)}$ almost surely.
2. $\lim_{n \rightarrow \infty} \frac{nd_n}{\ln(n)} = \frac{p}{2\lambda(1-p)}$ almost surely.

Proof. Let c_n^* and d_n^* be the connectivity and largest nearest neighbor distances respectively of $G_n^*(\lambda, \delta)$. Let $Y_{i,n}$ be the spacings between the vertices in G_n^* . The $Y_{i,n}$ are independent and exponentially distributed with mean $(\lambda(N - i))^{-1}$, where $N = \lfloor n/p \rfloor$, and $p = 1 - \exp(-\lambda T)$. Note that $(1 - p)/p \sim N/n - 1$. Let $y = y_n := (p(1 + \epsilon) \ln(n))/(n(1 - p)\lambda)$. We use the notation $f(n) \lesssim g(n)$ to mean that $f(n)$ is asymptotically bounded by a function $bh(n)$ where b is a constant and $h(n) \sim g(n)$.

Let $n_k = k^a$, be a subsequence with constant a to be chosen later. Let $N_k = \lfloor n_k/p \rfloor$.

$$\begin{aligned}
\Pr\left(\bigcup_{n=n_k}^{n_{k+1}} (c_n \geq y_n)\right) &\leq \Pr\left(c_{n_k} \geq \frac{p(1 + \epsilon) \ln(n_k)}{n_{k+1}(1 - p)\lambda}\right) \\
&\sim \Pr\left(c_{n_k}^* \geq \frac{p(1 + \epsilon) \ln(n_k)}{n_{k+1}(1 - p)\lambda}\right) \\
&= \Pr\left(\bigcup_{i=1}^{n_k-1} \left(Y_{i,n_k} \geq \frac{p(1 + \epsilon) \ln(n_k)}{n_{k+1}(1 - p)\lambda}\right)\right) \\
&\leq \sum_{i=1}^{n_k-1} \exp\left(-\frac{(N_k - i)p(1 + \epsilon) \ln(n_k)}{n_{k+1}(1 - p)}\right) \\
&= \sum_{j=N_k-n_k+1}^{N_k-1} \left(\frac{1}{n_k}\right)^{\frac{jp(1+\epsilon)}{n_k(1-p)}} \\
&< n_k \left(\frac{1}{n_k}\right)^{\frac{(N_k-n_k+1)p(1+\epsilon)}{n_k(1-p)}} \sim \left(\frac{1}{n_k}\right)^{\frac{(N_k/n_k-1)p(1+\epsilon)}{(1-p)}-1} \\
&\lesssim \frac{1}{k^{a\epsilon}},
\end{aligned}$$

where the last line follows by using the fact that $N_k/n_k \rightarrow p^{-1}$, as $k \rightarrow \infty$. Thus, for any $a > 1/\epsilon$, we get

$$\sum_{k=0}^{\infty} \Pr\left(\bigcup_{n=n_k}^{n_{k+1}} \left(c_n \geq \frac{p(1 + \epsilon) \ln(n)}{n(1 - p)\lambda}\right)\right) < \infty \quad \forall \epsilon > 0.$$

It follows from the Borel-Cantelli lemma that

$$\limsup_{n \rightarrow \infty} \frac{\lambda nc_n}{\ln(n)} \leq \frac{p}{1 - p} \quad a.s.$$

To establish the lower bound, we take $y_n = p(1 - \epsilon) \ln(n)/(n(1 - p)\lambda)$, and show that $\Pr(c_n^* \leq y_n)$ is summable;

$$\Pr(c_n^* \leq y_n) = \prod_{j=N-n+1}^{N-1} (1 - e^{-\lambda j y_n}).$$

Since $y_n \rightarrow 0$, and $ny_n \rightarrow \infty$, we have

$$\ln(\Pr(c_n^* \leq y_n)) \lesssim -\frac{(1-p)n}{p(1-\epsilon)\ln(n)},$$

and hence $\Pr(c_n^* \leq y_n)$ is summable. This completes the proof of the first part.

Proof for the largest nearest neighbor distance is similar. In the proof of the upper bound we take $y_n = p(1 + \epsilon) \ln(n)/(2n\lambda(1 - p))$ and use the following inequalities.

$$\begin{aligned} \Pr(d_n^* \geq y_n) &= \Pr\left(\bigcup_{i=2}^{n-1} ((Y_{i-1,n} \geq y_n) \cap (Y_{i,n} \geq y_n)) \cup (Y_{1,n} \geq y_n) \cup (Y_{n-1,n} \geq y_n)\right) \\ &\leq \sum_{i=2}^{n-1} \Pr(Y_{i-1,n} \geq y_n) \Pr(Y_{i,n} \geq y_n) + \Pr(Y_{1,n} \geq y_n) + \Pr(Y_{n-1,n} \geq y_n). \end{aligned}$$

To prove the lower bound set $y_n = p(1 - \epsilon) \ln(n)/(2n\lambda(1 - p))$ and proceed as follows.

$$\begin{aligned} \Pr(d_n^* \leq y_n) &= \Pr\left(\bigcap_{i=2}^{n-1} ((Y_{i-1,n} \leq y_n) \cup (Y_{i,n} \leq y_n)) \cap (Y_{1,n} \leq y_n) \cap (Y_{n-1,n} \leq y_n)\right) \\ &\leq \Pr\left(\bigcap_{i=1}^{\lfloor n/2 \rfloor} ((Y_{2i-1,n} \leq y_n) \cup (Y_{2i,n} \leq y_n))\right) \\ &\leq \prod_{i=1}^{\lfloor n/2 \rfloor} (1 - e^{-\lambda(2N-4i-1)y_n}). \end{aligned}$$

Take logarithms and use appropriate Taylor expansions and asymptotic equivalences as in the proof of the first part. We omit the details. This completes the proof of the Theorem. \square

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